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COMMENT

Potts model on Sierpinski carpets

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Abstract. Migdal-Kadanoff bond-moving renormalisation is used to study the q-state Potts model on Sierpinski carpets. A general approximate recursion relation including q as a parameter is given. The property of 'bond-interchanging invariance' is found and used in deriving the recursion relation. Fixed points and critical exponents for some carpets are presented. Marginal fixed points instead of unstable ones are found. Several typical flow diagrams are also shown.

The physical models on fractals with finite ramification, such as the Koch curve and Sierpinski gasket, exhibit no phase transition at finite temperature [1-3]. When the fractals possess infinite ramification the finite temperature transition is possible. Gefen *et al* have studied the Ising model on Sierpinski carpets by using Migdal-Kadanoff RSRG transformations [4-6]. They obtained a recursion relation and used it to find the fixed points and critical exponents for central cutout Sierpinski carpets [4]. We consider, on the same type of carpets, the Potts model.

We employ the same renormalisation scheme as Gefen *et al* [4] used for the Ising model (Migdal-Kadanoff bond-moving renormalisation) to construct the recursion relation for the Potts model. The relation required is

$$e^{-K_i'}=\frac{[1+(q-1)\,e^{-K_m}]^{b-l}[1+(q-1)\,e^{-\tilde{K}}]^l-(1-e^{-K_m})^{b-l}(1-e^{-\tilde{K}})^l}{[1+(q-1)\,e^{-K_m}]^{b-l}[1+(q-1)\,e^{-\tilde{K}}]^l+(q-1)(1-e^{-K_m})^{b-l}(1-e^{-\tilde{K}})^l}\qquad i=1,2 \tag{1}$$

where $K_1'=K'$ and $K_2'=K_w'$ are renormalised interaction constants [4]. The variables K_m and $\tilde{K}$ are defined by

$$\text{for } K' \qquad K_m=bK \qquad \tilde{K}=(b-l-1)K+2K_w \tag{2}$$

$$\text{for } K_w' \qquad K_m=\tfrac{1}{2}(b-1)K+K_w \qquad \tilde{K}=\tfrac{1}{2}(b-l-2)K+2K_w. \tag{3}$$

In (2) and (3), K and K_w are two types of interaction constant (as in [4]) on Sierpinski carpets and b and l are the structure parameters of the carpets: $l\times l$ subsquares are eliminated from $b\times b$ subsquares [4]. q is the number of states of the Potts model.

Recursion relation (1) is a general expression which not only includes the structure parameters b and l but is also a function of q. So it automatically treats the Ising model as the special case of $q=2$. It should be mentioned that relation (1) is an

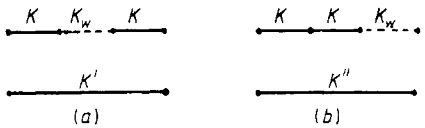

Figure 1. Two equivalent renormalisations: $K' = K''$.

approximate recursion because the bond-moving renormalisation used is itself an approximate method. We present the recursion (1) in the form of e^{-x} which is most convenient for the Potts model. The variable form $\tanh x$ is convenient for the Ising model but can hardly work on the Potts model.

In constructing the recursion relation we find and use the property of 'bond-interchanging invariance' explained as follows. We find that the following decimation renormalisations (figures 1(*a*) and (*b*)) are equivalent for any q value. With the help of these we constructed a RG procedure shown schematically in figure 2 which has been used in deriving recursion (1).

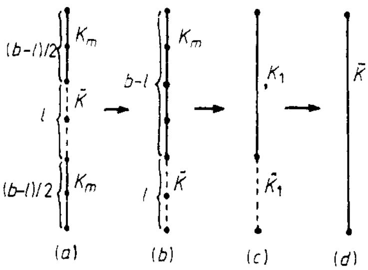

Figure 2. A renormalisation procedure (*a*) after bond moving. There are two sections of $(b-l)/2$ K_m bonds separated by a section of $l\,\tilde{K}$ bonds; (*b*) after bond interchanging, (*c*) making the first decimation and (*d*) final renormalised bond $\bar{K}$ after the second decimation.

Four matrix elements determining critical exponents can be easily derived from the recursion relation (1). They are

$$\left(\frac{\partial K_i'}{\partial K_j}\right) = \frac{q^2[\frac{2}{1}]^{b-l}[\frac{4}{3}]^l}{\{1-[\frac{2}{1}]^{b-l}[\frac{4}{3}]^l\}\{1+(q-1)[\frac{2}{1}]^{b-l}[\frac{4}{3}]^l\}}\left[\frac{(b-l)\,e^{-K_m}}{[1][2]}\left(\frac{\partial K_m}{\partial K_j}\right)\right.$$
$$\left.+\frac{l\,e^{-\tilde{K}}}{[3][4]}\left(\frac{\partial \tilde{K}}{\partial K_j}\right)\right] \qquad i,j=1,2. \tag{4}$$

Here we have used the notation

$$[1]=[1+(q-1)\,e^{-K_m}] \qquad [2]=[1-e^{-K_m}]$$
$$[3]=[1+(q-1)\,e^{-\tilde{K}}] \qquad [4]=[1-e^{-\tilde{K}}]. \tag{5}$$

The numerical results for critical points and critical exponents are summarised in table 1 where critical points are shown in coordinates (e^{-K}, e^{-K_w}). Fixed points E and F (see flow diagrams in figure 3) are non-trivial. Besides these there are three trivial fixed points for each carpet, A, B and C, with coordinates $(K, K_w) = (0, 0)$, $(0, \infty)$ and (∞, ∞) respectively. When $b = l+2$ an additional trivial point $D = (\infty, 0)$ appears.

In table 1 we see that point E is marginal, $y_K > 0$ and $y_{K_w} = 0$, and the y_{K_w} value is independent of q. Hence it also holds in the Ising model. Point F with $b = l+2 > 3$ is also marginal regardless of the value of q. These results differ from those of reference [4] where the points E and F are tricritical and critical points, respectively. In reference

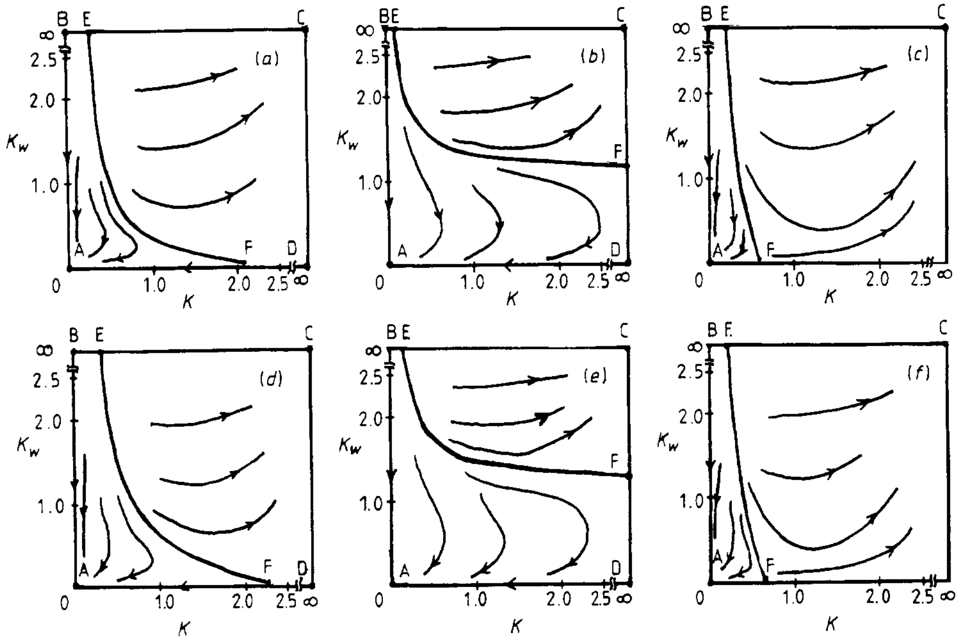

Figure 3. Flow diagrams for the Potts model on Sierpinski carpets with central cutout. (*a*) $(q, b, l) = (3, 3, 1)$, (*b*) $(q, b, l) = (3, 5, 3)$, (*c*) $(q, b, l) = (3, 7, 3)$, (*d*) $(q, b, l) = (4, 3, 1)$, (*e*) $(q, b, l) = (5, 5, 3)$, (*f*) $(q, b, l) = (4, 7, 3)$.

[4] $(\partial \tanh K_i'/\partial \tanh K_j)$ is used in calculating critical exponents, $(\partial \tanh K_i'/\partial \tanh K_j) = (\cosh^2 K_j/\cosh^2 K_i')(\partial K_i'/\partial K_j)$. We guess that in the calculations of reference [4] $(\cosh^2 K_i/\cosh^2 K_i')$ is not taken to be unity at the infinite fixed point. This accounts for the differences in critical exponents between this paper and reference [4]. In our calculation the matrix elements $(\partial K_i'/\partial K_j)$ are used according to the definition of critical exponents.

We conjecture that the marginal critical points (K^*, ∞) and (∞, K_w^*) are due to the peculiarity of elimination. A carpet with $K_m = \alpha K + \beta K_w$ (for K') instead of the case $K_m = bK$ (see (2)) will shift the critical point (K^*, ∞) to (K^*, K_w^*). Similarly, the critical point (∞, K_w^*) when $b = l + 2 > 3$ is the result of $\tilde{K} = 2K_w$ (for K_w', see (3)); if an elimination of a carpet makes $\tilde{K} = \alpha' K + \beta' K_w$, (∞, K_w^*) would disappear. All these deserve further study.

Taking $q = 2$ we repeat Gefen *et al*'s results for the Ising model [4] except for the marginal case $y = 0$ pointed out above (see table 1). The critical points of the $q = 2$ Potts model obey the exact relation $K_{\mathrm{Potts}}^* = 2K_{\mathrm{Ising}}^*$ [7] with the Ising model. Our numerical results agree with the relation. Table 1 lists the exponents of $q = 5$ for we expect that the critical q value q_c (beyond which the transition becomes first order) on Sierpinski carpets will be larger than four. There are a few calculating mistakes in reference [4]: $(b, l) = (3, 1)$, $\lambda_{F_2} = -0.137$ not -0.797; $(b, l) = (15, 7)$, $D = 1.909$ not 1.329; $(b, l) = (15, 11)$, $D = 1.715$ not 1.716.

The dependence of critical exponents on D (i.e. b and l) and q can be seen from table 1. We see that, except for the case $b = 3$, $l = 1$, y_t (the thermal exponent represented by the value of y larger than zero from two values of y) decreases with decreasing D at a constant q, which includes l increasing for b fixed and b decreasing for l fixed. This conclusion has been reached by Gefen *et al* [4] for the Ising model. Now we see

Table 1. Results for the Potts model on central cutout Sierpinski carpets.

				Point E				Point F			
q	b	l	D	e^{-K}	e^{-K_w}	y_K	y_{K_w}	e^{-K}	e^{-K_w}	y_{F_1}	y_{F_2}
2	3	1	1.893	0.786	0	0.562	0	0.177	0.969	0.224	−0.137
2	7	5	1.633	0.959	0	0.349	0	0	0.203	0	0.342
2	11	9	1.538	0.984	0	0.287	0	0	0.112	0	0.286
2	7	1	1.989	0.777	0	0.683	0	0.739	0.996	0.694	−1.873
2	7	3	1.986	0.841	0	0.594	0	0.577	0.991	0.524	−1.445
2	11	3	1.968	0.843	0	0.648	0	0.779	0.999	0.652	−2.089
2	31	29	1.394	0.998	0	0.202	0	0	0.0345	0	0.202
2	31	23	1.767	0.954	0	0.507	0	0.608	0.999 95	0.476	−2.128
3	3	1	1.893	0.746	0	0.619	0	0.124	0.983	0.160	−0.106
3	5	3	1.723	0.899	0	0.448	0	0	0.319	0	0.423
3	5	1	1.975	0.720	0	0.738	0	0.628	0.993	0.743	−1.906
3	7	1	1.989	0.751	0	0.747	0	0.713	0.998	0.758	−2.188
3	7	3	1.986	0.815	0	0.651	0	0.545	0.995	0.561	−1.724
3	7	5	1.633	0.946	0	0.373	0	0	0.190	0	0.371
3	15	1	1.998	0.836	0	0.724	0	0.830	0.999 97	0.730	−2.918
3	15	5	1.957	0.859	0	0.683	0	0.788	0.9999	0.672	−2.690
3	15	7	1.909	0.875	0	0.650	0	0.719	0.9998	0.601	−2.318
3	15	9	1.835	0.897	0	0.599	0	0.572	0.999	0.533	−1.905
3	31	29	1.374	0.997	0	0.206	0	0	0.0340	0	0.206
4	3	1	1.893	0.717	0	0.662	0	0.0952	0.989	0.123	−0.0864
4	7	1	1.989	0.733	0	0.793	0	0.694	0.999	0.803	−2.443
4	7	3	1.986	0.796	0	0.692	0	0.522	0.996	0.588	−1.946
4	7	5	1.633	0.935	0	0.392	0	0	0.180	0	0.393
5	3	1	1.893	0.694	0	0.695	0	0.0773	0.993	0.0992	−0.0732
5	5	1	1.975	0.679	0	0.823	0	0.588	0.996	0.823	−2.312
5	5	3	1.723	0.867	0	0.497	0	0	0.285	0	0.475

it also holds on the Potts model. In all non-($b=3$, $l=1$) cases, y_t increases with q monotonically. We find a similar result in normal lattices [8, 9].

As has been pointed out by reference [4] $b=3$, $l=1$ is a special case. It satisfies $b=l+2$ but differs from all $b=l+2>3$ cases in the locations of critical points, values of critical exponents and the patterns of flow diagrams. Now the variation of the exponents with q in the $b=3$, $l=1$ case is also unusual: at point E $y_K(=y_t)$ increases with q but at point F, $y_{F_1}(=y_t)$ decreases as q increases.

We show the flow diagrams in (K, K_w) space. In each diagram the critical line EF separates the ordered phase (upper part) from the disordered one (lower part). The flow diagrams in figure 3 can be divided into three classes: $b=l+2=3$, $b=l+2>3$ and $b>l+2$ according to the locations of the critical points. This classification is independent of q and valid for the Ising model. In fact Gefen *et al* have made the same classification in their work for the Ising model [4]. We see that when q varies the patterns of flow diagrams remain unchanged. However, the ordered phase area reduces as q increases (see figure 3).

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